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Constant savings rates and quasi-arithmetic population growth under exhaustible resource constraints

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Abstract

In the Dasgupta–Heal–Solow–Stiglitz (DHSS) model of capital accumulation and resource depletion we show the following equivalence: if an efficient path has constant (gross and net of population growth) savings rates, then population growth must be quasi-arithmetic and the path is a maximin or a classical utilitarian optimum. Conversely, if a path is optimal according to maximin or classical utilitarianism (with constant elasticity of marginal utility) under quasi-arithmetic population growth, then the (gross and net of population growth) savings rates converge asymptotically to constants.

Keywords: Quasi-arithmetic population growth; Constant savings rates; Exhaustible resources; Sustainability

1. Introduction

In this paper we revisit a question posed by Mitra [17]: What patterns of population growth are consistent with the attainment of some well-known social objectives (i.e., maximin and classical utilitarianism) in the presence of exhaustible resource constraints? Prior to Mitra's investigation it was known—as shown by Solow [22] and Stiglitz [23]—that non-decreasing per capita consumption is infeasible under *exponential* population growth when exhaustible resources are essential inputs in production and there is no technological progress. Mitra [17], however, established that non-decreasing per capita consumption is feasible under *quasi-arithmetic*

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population growth[1] in a discrete time version of the Cobb–Douglas Dasgupta–Heal–Solow–Stiglitz (DHSS) model of capital accumulation and resource depletion [5,22,23].

Mitra [17] analyzed this question without imposing a specific parametric structure on population growth, while considering quasi-arithmetic growth in examples. In this paper we aim for explicit closed form solutions, since we believe that this is essential for the applicability of the results. Hence, we concentrate on the case of quasi-arithmetic population growth, leading to growth paths that are *regular* in the terminology of Groth et al. [9]. To further facilitate such tractability, we consider the original *continuous time* version of the Cobb–Douglas DHSS model. It is well-known that the Cobb–Douglas production function is of particular interest in the context of the DHSS model since—in the case of no population growth and no technological progress—it is the only CES specification that allows for non-decreasing per capita consumption without making the resource inessential.

We illustrate in this paper the feasibility of paths with non-decreasing per capita consumption in spite of population growth by presenting closed-form solutions. In contrast to Mitra [17], we also include the case of population decline. *This paper substantially extends Mitra's analysis by showing the equivalence between efficiency and* constant (*gross and net of population growth*) *savings rates*, *on the one hand*, *and quasi-arithmetic population growth and the social objectives of maximin and classical* (*undiscounted*) *utilitarianism*, *on the other hand*. Both maximin and classical utilitarianism treat generations equally and fit many people's view of an intergenerationally equitable social objective much better than discounted utilitarianism. From this perspective it is of interest to note that this paper highlights cases where positive discounting is not needed to ensure the existence of a socially optimal choice, since both maximin and utilitarianism with zero discounting give sensible and interesting results.

In a neglected contribution that is a precursor to the present paper, Hoel [15] provides an early analysis of constant savings rates in the Cobb–Douglas DHSS model.[2] He characterizes paths arising from constant savings rates when there is no technological progress, but does not discuss the optimality of such paths and does not consider population growth. Conversely, Solow [22] and Stiglitz [23] (in the case of maximin) and Dixit [7, pp. 169–171] and Dasgupta and Heal [6, pp. 303–308] (in the case of classical utilitarianism) show that optimal growth paths may exhibit constant savings rates in the Cobb–Douglas DHSS model, although they (with the exception of Dixit) do not emphasize this property. Recently, paths with constant savings rates in this particular model have attracted some attention [2,13,19]. *This paper presents a complete characterization of constant savings rate paths in a setting with population growth—but without technological progress—and emphasizes their relationship to the social objectives of maximin and classical utilitarianism.*

In the Cobb–Douglas DHSS model, the Hartwick rule—which there takes the form of prescribing that resource rents be reinvested in reproducible capital—entails a constant savings rate equaling the constant relative functional share of resource input. An efficient path that develops according to the Hartwick rule in a setting where there is no population growth and no technological progress attains constant consumption and is a maximin optimum. Moreover, since Hartwick's original contribution [14], there has been much interest in the converse result: whether a maximin objective leads to paths following the Hartwick rule, and thus having a constant savings rate in this particular model [8,26,18,27,4]. *This paper generalizes the literature on the Hartwick rule and its converse*, *by considering also the case where population growth is non-zero and by including also classical utilitarianism as an objective.*[3]

Due to the Cobb–Douglas production function, the relative functional share of capital is constant. It turns out to be a necessary condition for the existence of paths with constant savings rates that the gross of population growth savings rate is smaller than the relative functional share of capital. This means that the functional share of capital must not only cover the accumulation of per capita capital, but also the "drag" on per capita capital accumulation caused by population growth. *This paper thereby generalizes a well-known*

[1] See Definition 3 of Section 2.2 for the definition of quasi-arithmetic population growth.

[2] Constant savings rate paths are also studied by Stiglitz [24] and Dixit [7, Chapter 7] in the case of exponential growth in technology and population.

[3] Even though a path developing according to the Hartwick rule in the Cobb–Douglas DHSS model has a constant savings rate, we refrain from referring to other constant savings rate paths as paths following a "generalized" Hartwick's rule. The reason is that the term "generalized Hartwick rule" has already been given a different meaning by Dixit et al. [8], namely that the present value of net investments is constant; see also [11,3,12].

condition for the feasibility of positive constant consumption, shown by Solow [22] and Stiglitz [23] in the case of no population growth and no technological progress.[4]

The paper is organized as follows. In Section 2 we introduce the model and present preliminary results. In Section 3 we show that if an efficient path has constant (gross and net of population growth) savings rates, then population growth is quasi-arithmetic and the path is a maximin or classical utilitarian optimum. In Section 4 we then establish a converse result: if a feasible path is optimal according to maximin or classical utilitarianism (with constant elasticity of marginal utility) under quasi-arithmetic population growth, then the (gross and net of population growth) savings rates converge to constants asymptotically. In Section 5 we consider quasi-arithmetic technological progress and show that the implications of this are similar but not identical to quasi-arithmetic population decline. In Section 6 we end by offering concluding remarks.

2. The setting

2.1. The model

Consider the Cobb–Douglas version of the DHSS model:

$$Q = AK^{\alpha}R^{\beta}N^{1-\alpha-\beta} = C + I,$$

where we denote by Q non-negative net production, by A positive state of technology, by K non-negative capital, by R non-negative resource input, by N positive population, and by C non-negative consumption, and where $I := \dot{K}$ and

$$\alpha > 0, \quad \beta > 0, \quad \alpha + \beta < 1.$$

The assumption that $\alpha + \beta < 1$ means that labor inputs are productive. Most results hold also if $\alpha + \beta = 1$. Let the lower-case variables, q, c, k, r, i, refer to per capita values so that

$$q = Ak^{\alpha}r^{\beta} = c + i = c + \frac{\dot{N}}{N}k + \dot{k}. \tag{1}$$

For exogenously given absolutely continuous paths of the state of technology and population, $\{A(t)\}_{t=0}^{\infty}$ and $\{N(t)\}_{t=0}^{\infty}$, and positive initial stocks of capital and resource, $(K_0, S_0) \gg 0$, the path $\{q(t), c(t), k(t), r(t)\}_{t=0}^{\infty}$ is *feasible* if

$$N(0)k(0) = K_0, \tag{2}$$

$$\int_0^{\infty} N(t)r(t)\,dt \leqslant S_0 \tag{3}$$

are satisfied, and (1) holds for almost every $t > 0$. We assume that $\{k(t)\}_{t=0}^{\infty}$ is absolutely continuous and that $\{q(t)\}_{t=0}^{\infty}$, $\{c(t)\}_{t=0}^{\infty}$, and $\{r(t)\}_{t=0}^{\infty}$ are piecewise continuous cf. [20, pp. 72–73]. Henceforth, *a "path" will always refer to a feasible path*. A path $\{q(t), c(t), k(t), r(t)\}_{t=0}^{\infty}$ is *interior* if $(q(t), c(t), k(t), r(t)) \gg 0$ for almost every $t > 0$.

Denote by $v(t) := \dot{N}(t)/N(t)$ the rate of population growth. For an interior path, denote by $x(t) := k(t)/q(t)$ the capital-output ratio, and by $z(t) := v(t)x(t)$ the "drag" on capital accumulation caused by population growth. Then

$$a(t) := \frac{i(t)}{q(t)},$$

$$b(t) := \frac{\dot{k}(t)}{q(t)} = a(t) - z(t) \tag{4}$$

are the *gross of population growth* and *net of population growth savings rates*, respectively (where the last equality in (4) follows from (1)).

[4] Stiglitz [23] obtains the same result also for "steady state paths" in the case of exponential population growth and exponential technological progress.

2.2. Definitions

In Sections 3 and 4 we show the equivalence between efficiency and constant (gross and net of population growth) savings rates, on the one hand, and quasi-arithmetic population growth and the social objectives of maximin and classical utilitarianism, on the other hand. In this subsection we formally define these concepts.

Definition 1. The economy has *constant gross of population growth savings rate* if $a(t) = a^*$, a constant, for all $t > 0$.

Definition 2. The economy has *constant net of population growth savings rate* if $b(t) = b^*$, a constant, for all $t > 0$.

Definition 3. Population growth is *quasi-arithmetic* if $N(t) = N(0)(1 + \mu t)^{\varphi}$ for all $t \geqslant 0$, where $\mu > 0$ and φ are constants.

Definition 4. A path $\{q(t), c(t), k(t), r(t)\}_{t=0}^{\infty}$ is optimal under a *maximin* objective if $\inf_{t\geqslant 0} c(t) > 0$ and

$$\inf_{t\geqslant 0} c(t) \geqslant \inf_{t\geqslant 0} \bar{c}(t)$$

for any path $\{\bar{q}(t), \bar{c}(t), \bar{k}(t), \bar{r}(t)\}_{t=0}^{\infty}$.

Definition 5. A path $\{q(t), c(t), k(t), r(t)\}_{t=0}^{\infty}$ is optimal under a *classical utilitarian* objective with utility function u if

$$\lim \sup_{T\to\infty} \left(\int_0^T N(t)u(\bar{c}(t))\,dt - \int_0^T N(t)u(c(t))\,dt \right) \leqslant 0$$

for any path $\{\bar{q}(t), \bar{c}(t), \bar{k}(t), \bar{r}(t)\}_{t=0}^{\infty}$.

Definition 1 implies the savings rule $\dot{K} = a^*Q$, while Definition 2 corresponds to the savings rule $d(K/N)/dt = b^*Q/N$. With a constant population, the gross and net of population growth savings rates coincide. In this case, the assumption of a constant savings rate has a long tradition in the growth-theoretic literature. Note in particular that, without population growth and with a constant savings rate, the model described above coincides with a simple Solow–Swan model [21,25] if $\beta = 0$. In the Cobb–Douglas version of the DHSS model, Hoel [15] presented an early analysis of the assumption of a constant savings rate (which does not necessarily equal β).

Definition 3 includes the cases where population grows ($\varphi > 0$), is constant ($\varphi = 0$), and declines ($\varphi < 0$). With a growing population, it follows from Definition 3 that population is a convex (concave) function of time if $\varphi > 1$ ($0 < \varphi < 1$). In either case, population increases beyond all bounds, while the rate of population growth is a hyperbolic function of time, approaching zero as time goes to infinity. Quasi-arithmetic population growth as defined in Definition 3 may be a better approximation than exponential growth to the future development of the world's population, now that the global population growth rate is declining. In fact, global population change since 1990 indicates that the absolute increase in population is also decreasing. This means that global population is experiencing sub-arithmetic growth, suggesting that $0 < \varphi < 1$.

Definition 4 entails that a maximin optimum is non-trivial in the sense of maintaining a positive per capita consumption level. When applying the classical utilitarian objective of Definition 5, we will assume constant elasticity of marginal utility:

$$u(c) = c^{1-\eta}/(1-\eta),$$

where $\eta > 0$, with $\eta = 1$ corresponding to the case where $u(c) = \ln c$.

2.3. Sufficient and necessary conditions for efficiency

A path is *efficient* if there is no path with at least as much consumption everywhere and larger consumption on a subset of the time interval with positive measure. An interior path satisfies the *Hotelling rule* if the inverse of the marginal productivity of resource input

$$p(t) := \frac{1}{\beta \frac{q(t)}{r(t)}} \tag{5}$$

is absolutely continuous and, for almost every $t > 0$,

$$\alpha \frac{q(t)}{k(t)} = -\frac{\dot{p}(t)}{p(t)}. \tag{6}$$

The Hotelling rule ensures no profitable arbitrage of resource input, and implies that $\{q(t)\}_{t=0}^{\infty}$ and $\{r(t)\}_{t=0}^{\infty}$ are absolutely continuous. A path satisfies *resource exhaustion* if (3) is binding. A path satisfies the *capital value transversality condition* if

$$\lim_{t \to \infty} p(t)N(t)k(t) = 0. \tag{7}$$

The following results provide sufficient and necessary conditions for the efficiency of interior paths. The sufficiency result builds on Malinvaud [16].

Lemma 6. *Let $\{q(t), c(t), k(t), r(t)\}_{t=0}^{\infty}$ be an interior path. The path $\{q(t), c(t), k(t), r(t)\}_{t=0}^{\infty}$ is efficient if it satisfies the Hotelling rule, resource exhaustion, and the capital transversality condition.*

Proof. By (5) and the fact that the path satisfies the Hotelling rule, it follows that $-\dot{p}/p = \alpha q/k = \alpha Q/K$ (i.e., the marginal product of capital) and $1/p = \beta q/r = \beta Q/R$ (i.e., the marginal product of resource input). Hence, if $\bar{Q} = A\bar{K}^{\alpha}\bar{R}^{\beta}N^{1-\alpha-\beta}$, the concavity of the production function implies that

$$Q + \frac{\dot{p}}{p}K - \frac{1}{p}R \geqslant \bar{Q} + \frac{\dot{p}}{p}\bar{K} - \frac{1}{p}\bar{R},$$

which can be rewritten (using $Q = C + \dot{K}$ and $\bar{Q} = \bar{C} + \dot{\bar{K}}$) as

$$p(\bar{C} - C) \leqslant -\frac{d}{dt}(p(\bar{K} - K)) + \bar{R} - R.$$

Let $\{\bar{q}(t), \bar{c}(t), \bar{k}(t), \bar{r}(t)\}_{t=0}^{\infty}$ be any path. Then, for all $T > 0$, (by integrating and using $K(0) = \bar{K}(0) = K_0$, $C = Nc$, $\bar{C} = N\bar{c}$, $K = Nk$, $\bar{K} = N\bar{k}$, $R = Nr$, and $\bar{R} = N\bar{r}$)

$$\int_0^T p(t)N(t)(\bar{c}(t) - c(t))\,dt \leqslant p(T)N(T)(k(T) - \bar{k}(T)) + \int_0^T N(t)(\bar{r}(t) - r(t))\,dt. \tag{8}$$

It follows that $\{q(t), c(t), k(t), r(t)\}_{t=0}^{\infty}$ is efficient since it satisfies resource exhaustion and (7), while $\{\bar{r}(t)\}_{t=0}^{\infty}$ satisfies (3) and $pN\bar{k}$ is non-negative. □

Lemma 7. *Let $\{q(t), c(t), k(t), r(t)\}_{t=0}^{\infty}$ be an interior path. If the path $\{q(t), c(t), k(t), r(t)\}_{t=0}^{\infty}$ is efficient, then it satisfies the Hotelling rule and resource exhaustion.*

Proof. Suppose $\int_0^{\infty} N(t)r(t)\,dt < S_0$. This obviously contradicts the efficiency of $\{q(t), c(t), k(t), r(t)\}_{t=0}^{\infty}$. Hence, the path satisfies resource exhaustion, and it also solves the so-called *minimum resource use problem*, i.e., for any path $\{\bar{q}(t), c(t), \bar{k}(t), \bar{r}(t)\}_{t=0}^{\infty}$ we have $\int_0^{\infty} N(t)\bar{r}(t)\,dt \geqslant \int_0^{\infty} N(t)r(t)\,dt$. The Hamiltonian of the minimum resource use problem reads

$$H(c, k, r, t, \lambda) = -Nr + \lambda(Ak^{\alpha}r^{\beta} - c - v(t)k).$$

The problem has an interior solution $\{q(t), c(t), k(t), r(t)\}_{t=0}^{\infty}$ with $\{k(t)\}_{t=0}^{\infty}$ being absolutely continuous and $\{r(t)\}_{t=0}^{\infty}$ and $\{c(t)\}_{t=0}^{\infty}$ being piecewise continuous. Hence, among the necessary conditions we have that $\{\lambda(t)\}_{t=0}^{\infty}$ is absolutely continuous and

$$\frac{\partial H}{\partial r} = 0 \quad \text{and} \quad \frac{\partial H}{\partial k} = -\dot{\lambda},$$

from which the Hotelling rule follows by setting $\lambda(t) = p(t)N(t)$. □

2.4. Sufficient conditions for optimality

An interior path satisfies the *Ramsey rule* if $\{c(t)\}_{t=0}^{\infty}$ is absolutely continuous and, for almost every $t>0$,

$$\eta\frac{\dot{c}(t)}{c(t)} = \alpha\frac{q(t)}{k(t)}, \tag{9}$$

recalling our assumption that the elasticity of marginal utility is constant. The Ramsey rule ensures no welfare enhancing arbitrage of consumption under classical utilitarianism.

The following result provides sufficient conditions for the optimality of interior paths.

Lemma 8. *Let $\{q(t), c(t), k(t), r(t)\}_{t=0}^{\infty}$ be an interior path that satisfies the Hotelling rule, resource exhaustion, and the capital transversality condition. If $\{q(t), c(t), k(t), r(t)\}_{t=0}^{\infty}$ has constant per capita consumption, then it is the unique maximin optimum. If $\{q(t), c(t), k(t), r(t)\}_{t=0}^{\infty}$ satisfies the Ramsey rule, then it is the unique classical utilitarian optimum.*

Proof. *Maximin optimum*: Let $\{q(t), c(t), k(t), r(t)\}_{t=0}^{\infty}$ be an interior path satisfying the Hotelling rule, resource exhaustion, and the capital transversality condition. By Lemma 6, $\{q(t), c(t), k(t), r(t)\}_{t=0}^{\infty}$ is efficient. If the path has constant consumption, then $\inf_{t\geqslant 0} c(t)>0$ since the path is interior, and $\inf_{t\geqslant 0} c(t)\geqslant\inf_{t\geqslant 0} \bar{c}(t)$ for any path $\{\bar{q}(t), \bar{c}(t), \bar{k}(t), \bar{r}(t)\}_{t=0}^{\infty}$ since the path is efficient.

Classical utilitarian optimum: Let $\{q(t), c(t), k(t), r(t)\}_{t=0}^{\infty}$ be an interior path satisfying the Hotelling rule, resource exhaustion, the capital transversality condition, and the Ramsey rule. Then

$$\eta\frac{\dot{c}(t)}{c(t)} = -\frac{\dot{p}(t)}{p(t)}$$

for almost every $t>0$, and we obtain

$$c(t)^{-\eta} = \lambda_0 p(t)$$

for all $t\geqslant 0$ by setting $\lambda_0 = c(0)^{-\eta}/p(0)$. Hence, with $u(c) = c^{1-\eta}/(1-\eta)$ and $\eta>0$ ($\eta = 1$ corresponding to $u(c) = \ln c$), $u(\bar{c}(t)) - u(c(t)) \leqslant \lambda_0 p(t)(\bar{c}(t) - c(t))$ for all $t\geqslant 0$, and any $\{\bar{c}(t)\}_{t=0}^{\infty}$. It now follows from the proof of Lemma 6 that

$$\limsup_{T\to\infty}\left(\int_0^T N(t)(u(\bar{c}(t)) - u(c(t))\,dt\right) \leqslant \lambda_0 \limsup_{T\to\infty}\left(\int_0^T p(t)N(t)(\bar{c}(t) - c(t))\,dt\right) \leqslant 0$$

for any path $\{\bar{q}(t), \bar{c}(t), \bar{k}(t), \bar{r}(t)\}_{t=0}^{\infty}$.

Uniqueness: Follows from the strict concavity of the production function w.r.t. K and R. I.e., the inequality in (8) is strict if $\{\bar{q}(t), \bar{c}(t), \bar{k}(t), \bar{r}(t)\}_{t=0}^{\infty}$ differs from $\{q(t), c(t), k(t), r(t)\}_{t=0}^{\infty}$ on a subset of $[0, T]$ with positive measure. □

3. Sufficiency of constant savings rates

In this section we explore the properties of efficient paths with constant savings rates in the case of a stationary technology (setting $A(t) = 1$ for all $t>0$). We establish the following two theorems.

Theorem 9. *There exists an interior and efficient path with constant gross of population savings rate, a, and a constant net of population savings rate, b, if and only if $\alpha>a$ and population growth is quasi-arithmetic with*

$$\mu = \sigma[(\alpha - a)^{\beta} K_0^{\alpha-1} S_0^{\beta} N(0)^{1-\alpha-\beta}]^{\frac{1}{1-\beta}}, \tag{10}$$

$$\varphi = \frac{a-b}{\sigma}, \tag{11}$$

where

$$\sigma = \frac{(1-\alpha-\beta)b + \alpha\beta}{1-\beta}. \tag{12}$$

Theorem 10. *If an interior and efficient path has constant gross of population savings rate, a, and a constant net of population savings rate, b, then the path is optimal under a maximin objective if $b = \beta$ and optimal under a* *classical utilitarian objective with constant elasticity of marginal utility given by*

$$\eta = \frac{1-\beta}{b-\beta} \tag{13}$$

if $b > \beta$.

It will turn out to be useful to rearrange (12) as follows:

$$(1-\beta)(b-\sigma) = \alpha(b-\beta). \tag{14}$$

To prove Theorems 9 and 10, we first report a proposition.

Proposition 11. *For an interior path satisfying the Hotelling rule, the following holds*:

(a) *The time derivative of the capital-output ratio*, x, *exists almost everywhere and equals*

$$\dot{x}(t) = \frac{(1-\alpha-\beta)b(t) + \alpha\beta}{1-\beta}. \tag{15}$$

(b) *If the path has a constant net of population growth savings rate*, b, *then the capital-output ratio is an affine function of time*

$$x(t) = x(0) + \sigma t = x(0)(1+\mu t), \tag{16}$$

where σ *is given by* (12) *and*

$$\mu = \frac{\sigma}{x(0)} = \sigma\frac{q(0)}{k(0)} = \sigma k(0)^{\alpha-1} r(0)^{\beta}. \tag{17}$$

(c) *If the path has constant gross of population growth savings rate*, a, *and constant net of population growth savings rate*, b, *then*

(i) *the path has quasi-arithmetic population growth with* φ *given by* (11),

(ii) *per capita output*, *consumption*, *capital stock and resource input are given by*

$$q(t) = q(0)(1+\mu t)^{\frac{b}{\sigma}-1}, \tag{18}$$

$$c(t) = (1-a)q(0)(1+\mu t)^{\frac{b}{\sigma}-1}, \tag{19}$$

$$k(t) = \frac{K_0}{N(0)}(1+\mu t)^{\frac{b}{\sigma}}, \tag{20}$$

$$r(t) = r(0)(1+\mu t)^{-\frac{\alpha-b}{\sigma}-1}, \tag{21}$$

Proof of Proposition 11. *Part* (a): Since the path satisfies the Hotelling rule, $\{q(t)\}_{t=0}^{\infty}$ and $\{r(t)\}_{t=0}^{\infty}$ are absolutely continuous. Feasibility (Eq. (1)) implies

$$\frac{\dot{q}}{q} = \alpha\frac{\dot{k}}{k} + \beta\frac{\dot{r}}{r}. \tag{22}$$

The Hotelling rule (Eq. (6)) implies

$$\alpha\frac{q}{k} = \frac{\dot{q}}{q} - \frac{\dot{r}}{r}. \tag{23}$$

By eliminating $\dot{r}/r$ from (22) and (23) and rearranging, we obtain

$$\frac{k}{q}\left(\frac{\dot{k}}{k} - \frac{\dot{q}}{q}\right) = \frac{(1-\alpha-\beta)\frac{\dot{k}}{q} + \alpha\beta}{1-\beta}.$$

Since, by the definition of x,

$$\dot{x} = \frac{d}{dt}\left(\frac{k}{q}\right) = \frac{k}{q}\left(\frac{\dot{k}}{k} - \frac{\dot{q}}{q}\right),$$

the result follows by applying (4).

Part (b): This follows from Part (a) through integration.

Part (c): Since $z(t) = v(t)x(t)$, it follows from (4) that

$$\frac{\dot{N}(t)}{N(t)} = v(t) = \frac{z(t)}{x(t)} = \frac{a-b}{x(t)}. \tag{24}$$

Hence,

$$N(t) = N(0)(1+\mu t)^{\frac{a-b}{\sigma}} \tag{25}$$

is obtained by solving (24) and applying (16), thus establishing (i).

Combining $q(t) = k(t)/x(t)$, $\dot{k}(t) = bq(t)$, (15), and (12) yields

$$\frac{\dot{q}(t)}{q(t)} = \frac{b-\sigma}{x(t)}. \tag{26}$$

By solving (26) and applying (16), we obtain (18). Furthermore, (19) follows from (18) and $c(t) = q(t) - i(t) = (1-a)q(t)$, while (20) follows from (2), (16), (18), and $k(t) = x(t)q(t)$. Note that it follows from (12) that $q(t)$ and $c(t)$ are increasing and $k(t)$ is a convex function of time if and only if $b > \beta$. Finally, since $q(t) = k(t)^{\alpha} r(t)^{\beta}$ and (by applying (14)) $(b/\sigma - 1 - \alpha b/\sigma)/\beta = -(\alpha - b)/\sigma - 1$, we obtain (21). □

By applying (14) to (18) and (19) we see that per capita output and consumption are growing when $b > \beta$. One can then intuitively regard the savings rate a as the sum of β to compensate for the value of exhaustible resource depletion, plus $a - b$ to compensate for population growth, plus $b - \beta$ to give growth in per capita output.

We are now in a position to prove Theorems 9 and 10.

Proof of Theorem 9 (*Necessity*). Assume the existence of a path $\{q(t), c(t), k(t), r(t)\}_{t=0}^{\infty}$ that is interior and efficient with a constant gross of population savings rate, a, and a constant net of population savings rate, b. Then, by Lemma 7, the path satisfies the Hotelling rule (so, by Proposition 11, the path is partially characterized by Eqs. (25), (18)–(21)) and resource exhaustion.

Resource exhaustion combined with (25) and (21) yield

$$N(0)r(0)\int_0^{\infty}(1+\mu t)^{-\frac{\alpha-a}{\sigma}-1}\,dt = S_0. \tag{27}$$

This entails $\alpha > a$ and implies

$$r(0) = \frac{\mu(\alpha - a)S_0}{\sigma N(0)}, \tag{28}$$

while it follows from (2) and (17) that

$$q(0) = \frac{\mu K_0}{\sigma N(0)}. \tag{29}$$

With $\alpha > a$, the parameter μ as given by (10) is determined by eliminating $r(0)$ from (17) and (28). In turn, this value of μ inserted in (29) determines $q(0)$, and inserted in (28) it determines $r(0)$, giving closed form solutions for (18)–(21).

Sufficiency: Assume that $\alpha > a$, and let population growth be quasi-arithmetic with μ and φ given by (10)–(12). It has already been demonstrated that, with $\alpha > a$ and such quasi-arithmetic population growth, there exists an interior path characterized by Eqs. (25), (18)–(21), and (28)–(29). This path satisfies resource exhaustion and has a constant gross of population savings rate, a, and a constant net of population savings rate, b. It remains to show that the path is efficient. Since

$$p(t) = \frac{1}{\beta\frac{q(t)}{r(t)}} = \frac{r(0)}{\beta q(0)}(1+\mu t)^{-\frac{\alpha}{\sigma}},$$

it follows from (18) and (20) that the Hotelling rule is satisfied and from (25) and (20) that the capital value transversality condition is satisfied. Hence, by Lemma 6 the constructed path is efficient. □

Proof of Theorem 10. By Theorem 9, the premise is not vacuous and any path satisfying the premise is characterized by $\alpha > a$, (10)–(12), (25), (18)–(21), and (28)–(29), and satisfies the Hotelling rule, resource exhaustion, and the capital transversality condition. We have two cases to consider.

Case 1: $b = \beta$. Since $b = \sigma = \beta$, it follows from (19) that per capita consumption is constant. Since the path satisfies the Hotelling rule, resource exhaustion, and the capital transversality condition, Lemma 8 implies that it is optimal under a maximin objective.

Case 2: $b > \beta$. Since $b > \sigma$, it follows from (19) that per capita consumption increases:

$$\frac{\dot{c}(t)}{c(t)} = \frac{b-\sigma}{x(t)}.$$

By (9) the Ramsey rule holds if η satisfies

$$\frac{\alpha}{\eta x(t)} = \frac{b-\sigma}{x(t)}.$$

By eliminating σ by means of (14), we obtain that η is given by (13). Since the path satisfies the Hotelling rule, resource exhaustion, and the capital transversality condition, Lemma 8 implies that it is optimal under a classical utilitarian objective with constant elasticity of marginal utility given by (13). □

In the special case of a constant population ($\varphi = 0$), the results of Theorems 9 and 10 have been reported elsewhere. Hoel [15] shows the result of Theorem 9 when $\varphi = 0$, or equivalently, $a = b$. Solow [22, Sections 9–10] and Stiglitz [23, Propositions 5a and b] show that $a = b = \beta$ corresponds to a maximin optimum, thereby establishing the maximin part of Theorem 10 when $\varphi = 0$. The utilitarian part of Theorem 10 with zero population growth is implied by the analysis of Dasgupta and Heal [6, pp. 303–308]. Also, with $\varphi = 0$, the formulae in Proposition 11(c)(ii) are the same as the constant-technology versions of formulae in Pezzey [19, p. 476], allowing for differences in notation.

The analysis of this section (see (18) and (20)) implies that per capita output is an increasing function of time and per capita capital is a convex function of time if $b > \beta$, corresponding to classical utilitarianism, while per capita output is constant and per capita capital is a linear function if $b = \beta$, corresponding to maximin. In either case, the capital-output ratio is a linear function of time (cf. (16)), and the growth rates of per capita output and capital approach zero as time goes to infinity.

The path described in Proposition 11(c) can be used to illuminate the meaning of the concept of a "genuine savings indicator" cf. [10] in the presence of population growth. "Genuine savings" must be zero along the constant per capita consumption path that is optimal under maximin. However, the value of changes in per capita stocks, $d(K/N)/dt + (1/p)d(S/N)/dt$, equals

$$\dot{k}(t) - \frac{1}{p(t)}r(t) - \frac{v(t)}{p(t)}\int_t^\infty r(\tau)\,d\tau = \left(1 - 1 - \frac{v(t)\sigma}{\mu(\alpha-a)}\right)\beta q(t) = -\frac{v(t)\sigma}{\mu(\alpha-a)}\beta q(t)$$

and is negative along the maximin path with positive quasi-arithmetic population growth, as there is no compensation for the spread of the remaining resource stock on more people. This illustrates the qualitative result obtained in Proposition 6 of Asheim [1], with the following intuitive interpretation: When the rate of population growth is decreasing, it is not necessary for the current generation to compensate fully for current population growth in order to ensure sustainable development.

Theorem 9 shows that the existence of an interior and efficient path with constant savings rates does not only imply that population growth is quasi-arithmetic, but also that the parameters of the exogenous population path satisfy (10)–(12). What happens if population growth is quasi-arithmetic, but without satisfying the strong parameter restrictions that (10)–(12) entail? This motivates the analysis of the next section, where we consider optimal paths that have quasi-arithmetic population growth satisfying a weak parameter restriction.

4. Necessity of constant savings rates

In this section we turn to a converse result that takes as its premise that paths have quasi-arithmetic population growth and are optimal under a maximin or classical utilitarian objective. We establish the following two theorems in the case of a stationary technology (setting $A(t) = 1$ for all $t > 0$). Theorem 12 presents conditions under which there exist paths having quasi-arithmetic population growth and being optimal under a maximin or classical utilitarian objective, thereby establishing that the premise is not vacuous. Theorem 13 shows that any such path has gross and net of population growth savings rates that converge asymptotically to constants.

Theorem 12. *Let population growth be quasi-arithmetic with*

$$-1 \leqslant \varphi < \frac{\alpha}{\beta} - 1. \tag{30}$$

There exists a unique path that is optimal under a maximin objective. There exists a unique path that is optimal under a classical utilitarian objective if the constant elasticity of marginal utility satisfies

$$\eta > \frac{(1-\beta) + (1-\alpha-\beta)\varphi}{\alpha - \beta(1+\varphi)}. \tag{31}$$

Theorem 13. *If a path has quasi-arithmetic population growth satisfying* (30) *and is optimal under a maximin objective or under a classical utilitarian objective with constant elasticity of marginal utility satisfying* (31), *then the path is interior and efficient, and the gross of population growth and net of population growth savings rates converge asymptotically to the constants*

$$a^* = \beta(1+\varphi) + \frac{(1-\beta) + (1-\alpha-\beta)\varphi}{\eta}, \tag{32}$$

$$b^* = \beta + \frac{1-\beta}{\eta}, \tag{33}$$

where $\eta = \infty$ corresponds to the maximin objective, and $\eta < \infty$ is the constant elasticity of marginal utility under the classical utilitarian objective.

In the case of rapid population decline (i.e., $\varphi < -1$), the resource is not essential: the initial stock of capital can give rise to positive and non-decreasing consumption without resource inputs. Hence, since our purpose is to study savings behavior under exhaustible resource constraints, we choose to exclude this case.

To prove Theorems 12 and 13, we first report two propositions, in which we consider interior paths where the rate of per capita consumption growth is given by

$$\frac{\dot{c}(t)}{c(t)} = \frac{\alpha}{\eta x(t)}. \tag{34}$$

Eq. (34) includes the case of constant consumption by setting $\eta = \infty$.

Proposition 14. *Consider an interior path that satisfies the Hotelling rule and has quasi-arithmetic population growth with $\varphi \neq 0$. If the rate of per capita consumption growth given by* (34), *then the gross of population growth savings rate*, $a(t)$, *and the "drag" on capital accumulation caused by population growth*, $z(t)$, *are governed by*

$$\dot{a}(t) = \frac{\alpha(1 - a(t))v(t)}{(1 - \beta)z(t)}\left(a(t) - z(t) - \left(\beta + \frac{1 - \beta}{\eta}\right)\right), \tag{35}$$

$$\dot{z}(t) = \frac{(1 - \alpha - \beta)v(t)}{(1 - \beta)}\left(a(t) - \left(1 + \frac{1 - \beta}{(1 - \alpha - \beta)\varphi}\right)z(t) + \frac{\alpha\beta}{1 - \alpha - \beta}\right). \tag{36}$$

Proof. First, note that it follows from $\dot{k}(t) = b(t)q(t) = b(t)k(t)/x(t)$ that

$$\frac{\dot{k}(t)}{k(t)} = \frac{b(t)}{x(t)}. \tag{37}$$

Since $1 - a(t) = c(t)/q(t) = c(t)x(t)/k(t)$, it follows that

$$\begin{aligned}\frac{\dot{a}(t)}{1 - a(t)} &= -\frac{\dot{c}(t)}{c(t)} - \frac{\dot{x}(t)}{x(t)} + \frac{\dot{k}(t)}{k(t)} \\ &= \frac{\alpha}{(1 - \beta)x(t)}\left(b(t) - \left(\beta + \frac{1 - \beta}{\eta}\right)\right),\end{aligned}$$

where the last equation follows from (34), (15), and (37). Since $b(t) = a(t) - z(t)$ and $z(t) = v(t)x(t)$, we obtain (35) if $\varphi \neq 0$.

With quasi-arithmetic population growth, we have that $v(t) = \varphi\mu/(1 + \mu t)$ and

$$\frac{\dot{v}(t)}{v(t)} = -\frac{v(t)}{\varphi} \tag{38}$$

if $\varphi \neq 0$. Since $z(t) = v(t)x(t)$ it follows from (15) and (38) that

$$\dot{z}(t) = v(t)\left(\dot{x}(t) + \frac{\dot{v}(t)}{v(t)}x(t)\right) = v(t)\left(\frac{(1 - \alpha - \beta)b(t) + \alpha\beta}{1 - \beta} - \frac{v(t)x(t)}{\varphi}\right)$$

if $\varphi \neq 0$. Since $b(t) = a(t) - z(t)$ and $z(t) = v(t)x(t)$, we obtain (36). □

Proposition 15. *Let population growth be quasi-arithmetic with* $\varphi \neq 0$ *satisfying* (30) *and assume that* $\eta = \infty$ *or* η *satisfies* (31). *There exists a path satisfying resource exhaustion and Eqs.* (34)–(36), *and having the property that the gross of population growth and net of population growth savings rates converge asymptotically to the constants given by* (32) *and* (33). *This path is interior and satisfies the Hotelling rule and the capital value transversality condition.*

The proof of Proposition 15 is given in Appendix A. We are now in a position to prove Theorems 12 and 13.

Proof of Theorem 12. *Maximin. Case* 1: $\varphi = 0$. Consider the path characterized by (10)–(12), (25), (18)–(21), (28)–(29), and $a = b = \beta$. Since $0 = \varphi < \alpha/\beta - 1$ and $a = \beta$, so that $\alpha > a$, it follows from Theorem 9 that this zero population growth path exists. Furthermore, it is an interior path that satisfies the Hotelling rule, resource exhaustion, and the capital transversality condition, and has constant per capita consumption. By Lemma 8, it is the unique maximin optimum.

Case 2: $\varphi \neq 0$. The path established in Proposition 15 with $\eta = \infty$ is an interior path that satisfies the Hotelling rule, resource exhaustion, and the capital transversality condition, and has constant per capita consumption. By Lemma 8, it is the unique maximin optimum.

Classical utilitarianism. Case 1: $\varphi = 0$. Consider the path characterized by (10)–(12), (25), (18)–(21), (28)–(29), and $a = b = \beta + (1 - \beta)/\eta$. Since $0 = \varphi < \alpha/\beta - 1$, $a = \beta + (1 - \beta)/\eta$ and $\eta > (1 - \beta)/(\alpha - \beta)$, so that $\alpha > a$, it follows from Theorem 9 that this zero population growth path exists. Furthermore, it is an interior path that satisfies the Hotelling rule, resource exhaustion, the capital transversality condition, and the Ramsey rule. By Lemma 8, it is the unique classical utilitarian optimum.

Case 2: $\varphi \neq 0$. The path established in Proposition 15 with η satisfying (31) is an interior path that satisfies the Hotelling rule, resource exhaustion, the capital transversality condition, and the Ramsey rule. By Lemma 8, it is the unique classical utilitarian optimum. □

Proof of Theorem 13. By Theorem 12, there exists a unique optimal path, which is interior and (since it satisfies the Hotelling rule, resource exhaustion, and the capital transversality condition) efficient. By Proposition 15 and the proof of Theorem 12, the gross of population growth and net of population growth savings rates along this path converge asymptotically to the constants given by (32) and (33). □

5. Quasi-arithmetic technological progress

As shown by Pezzey [19], there exist constant savings rate paths also in the case where technological progress is quasi-arithmetic, while population is constant, provided that the quasi-arithmetic technological progress satisfies parameter restrictions. We include this case

- to provide a link between this paper's main results and Pezzey's [19] analysis,
- to demonstrate that such paths are maximin or classical utilitarian, and
- to point out that quasi-arithmetic technological progress does not correspond to quasi-arithmetic population decline.

Definition 16. Technological progress is *quasi-arithmetic* if $A(t) = A(0)(1 + \mu t)^{\theta}$ for all $t \geqslant 0$, where $\mu > 0$ and θ are constants.

The analysis of Groth et al. [9, Section 3] indicates that quasi-arithmetic technical progress may be just as plausible as exponential technological progress in the long run.

We establish the following result in the case of a constant population (setting $N(t) = 1$ for all $t > 0$). In this case, the gross and net of population growth savings rates coincide; therefore we denote by s the constant savings rate (where $s = a = b$). Also, since total and per capita values coincide, it follows that lower case variables also correspond to total net production, total consumption, total capital, and total resource input.

Theorem 17. *There exists an interior and efficient path with a constant savings rate, s, if $\alpha > s$ and technological progress is quasi-arithmetic with μ and θ satisfying*

$$(1 - \beta + \theta)\mu = ((1 - \alpha - \beta)s + \alpha\beta)[(\alpha - s)^{\beta} A(0) K_0^{\alpha-1} S_0^{\beta}]^{\frac{1}{1-\beta}}. \tag{39}$$

The path is optimal under a maximin objective if $s = \sigma$ and optimal under a classical utilitarian objective with constant elasticity of marginal utility given by

$$\eta = \frac{\alpha}{s - \sigma} \tag{40}$$

if $s > \sigma$, where

$$\sigma = \frac{(1 - \alpha - \beta)s + \alpha\beta}{1 - \beta + \theta}. \tag{41}$$

Proof. For the first part of the theorem, assume that $\alpha > s$, and let technological progress be quasi-arithmetic with μ and θ satisfying (39). With $\alpha > s$ and such quasi-arithmetic technological progress, there exists a path characterized by

$$q(t) = q(0)(1 + \mu t)^{\frac{s}{\sigma}-1}, \tag{42}$$

$$c(t) = (1 - s)q(0)(1 + \mu t)^{\frac{s}{\sigma}-1}, \tag{43}$$

$$k(t) = K_0(1 + \mu t)^{\frac{s}{\sigma}}, \tag{44}$$

$$r(t) = r(0)(1 + \mu t)^{-\frac{\alpha-s}{\sigma}-1}, \tag{45}$$

$$q(0) = \frac{\mu K_0}{\sigma}, \tag{46}$$

$$r(0) = \frac{\mu(\alpha - s)S_0}{\sigma}, \tag{47}$$

where σ is given by (41). To show this, take (42) as given. Then, (42) and $c(t) = q(t) - \dot{k}(t) = (1-s)q(t)$ imply (43). By letting the capital-output ratio $x(t) = k(t)/q(t)$ be given by

$$x(t) = x(0) + \sigma t = x(0)(1 + \mu t),$$

so that

$$\mu = \frac{\sigma}{x(0)} = \sigma \frac{q(0)}{K_0} = \sigma A(0) K_0^{\alpha-1} r(0)^{\beta}, \tag{48}$$

we obtain (44) and (46). Finally, (45) follows from $q(t) = A(t)k(t)^{\alpha} r(t)^{\beta}$ by applying (41), while (47) follows by, in addition, imposing resource exhaustion. By eliminating $r(0)$ from (47) and (48), it follows that the path exists if $\alpha > s$ and the parameters μ and θ satisfy (39).

The path is clearly interior. It remains to show that the path is efficient. Since

$$p(t) = \frac{1}{\beta \frac{q(t)}{r(t)}} = \frac{r(0)}{\beta q(0)} (1 + \mu t)^{-\frac{\alpha}{\sigma}},$$

it follows from (42) and (44) that the Hotelling rule is satisfied and from (44) that the capital value transversality condition is satisfied. Since, by construction, the path satisfies resource exhaustion, Lemma 6 implies that it is efficient.

For the second part of the theorem, we have two cases to consider.

Case 1: $s = \sigma$. It follows from (43) that per capita consumption is constant. Since the path satisfies the Hotelling rule, resource exhaustion, and the capital transversality condition, Lemma 8 implies that it is optimal under a maximin objective.

Case 2: $s > \sigma$. It follows from (43) that per capita consumption increases:

$$\frac{\dot{c}(t)}{c(t)} = \frac{s - \sigma}{x(t)}.$$

By (9) the Ramsey rule holds if η satisfies

$$\frac{\alpha}{\eta x(t)} = \frac{s - \sigma}{x(t)},$$

which implies (40). Since the path satisfies the Hotelling rule, resource exhaustion, and the capital transversality condition, Lemma 8 implies that it is optimal under a classical utilitarian objective with constant elasticity of marginal utility given by (40). □

The paths that Pezzey [19] considers satisfy the sufficient conditions of Theorem 17; this follows from straightforward but tedious calculations on the basis of his Eqs. (3)–(6) as well as the output expression on p. 476. Hence, it follows from Theorem 17 that Pezzey's paths are classical utilitarian in the case with increasing consumption, an observation not made by Pezzey [19].[5] Also Hoel [15] combines a constant savings rate with technological progress. But since he considers exponential technological progress, he obtains paths with different properties.

It follows from Eqs. (42)–(47) that the path $\{q(t), c(t), k(t), r(t)\}_{t=0}^{\infty}$ is as given by (18)–(21) and (28)–(29), except for the change in the definition of σ (compare (41) with (12)).

By (41), σ is increasing in s, with $\sigma = (1-\beta)/(1-\beta+\theta)$ if $s = \beta$. Hence, it follows from the proof of Theorem 17 that non-decreasing consumption is feasible even if less than all resource rents are reinvested (i.e., $s < \beta$), provided that there is quasi-arithmetic technological progress, since with $\theta > 0$ we may have that $\beta > s \geqslant \sigma$. The conditions $s \geqslant \sigma$ and (39) determine combinations of a constant savings rate and quasi-arithmetic technological progress that ensure non-decreasing consumption.

If $\theta = 0$, then the conditions $\alpha > s$ and $s \geqslant \sigma$ reduce to the well-known condition shown by Solow [21] and Stiglitz [23] for the Cobb–Douglas DHSS model in the case with no population growth and no technological

[5]Instead, Pezzey [19] shows optimality under discounted utilitarianism with a less concave utility function and a positive and decreasing discount rate. Since the discount rate is a hyperbolic function of *absolute* time, such a social objective is time-consistent, but not time-invariant.

progress, namely $\alpha > \beta$. However, if $\theta > 0$, then $\alpha > s$ and $s \geqslant \sigma$ are compatible with $\alpha < \beta$ since we may have that $\beta > s \geqslant \sigma$. Hence, non-decreasing consumption may be feasible even if $\alpha < \beta$.

The observations of the two previous paragraphs hold also in the case of a stationary technology and quasi-arithmetic population *decline*: (1) Non-decreasing per capita consumption may be feasible even if the gross of population growth savings rate a is smaller than β. (2) Non-decreasing per capita consumption may be feasible even if $\alpha < \beta$. However, by comparing the analyses of Sections 3 and 5 (in particular, observe that expressions (10)–(12) are different from expression (39)), it follows that the situation with a constant population and quasi-arithmetic technological progress is not a special case of the situation with a stationary technology and quasi-arithmetic population decline, or vice versa. Even though in the former situation net production can be expressed as a function of capital and resource input in efficiency units—corresponding in the latter situation to per capita net production being a function of per capita capital and per capital resource input—these two formulations do not lead to an identical expression for capital accumulation.

6. Concluding remarks

To highlight the findings of the present paper we will contrast it with the results obtained by Mitra [17]. He considers the same model, in discrete time, with a non-renewable natural resource and a Cobb–Douglas technology. However, he does not a priori specify any functional form for the population growth. He derives necessary and sufficient conditions for the existence of maximin and classical utilitarian optima. To illustrate, Mitra [17] employs quasi-arithmetic population growth and derives restrictions on the corresponding parameters satisfying these necessary and sufficient conditions. With this functional form for the population growth, his conditions coincide with those derived here: in the case of maximin, he states the conditions $\alpha > \beta$ (cf. condition (3.5a)) and $\varphi < (\alpha/\beta) - 1$ (stated in his Example 3.1); in the case of classical utilitarianism (with a constant elasticity of marginal utility), he states the conditions $\alpha > \beta$ (cf. condition (4.1a)), $\eta > (1-\beta)/(\alpha-\beta)$ (cf. condition (4.1b)), and

$$\varphi < \frac{\alpha - \beta - \dfrac{1-\beta}{\eta}}{\beta + \dfrac{1-\alpha-\beta}{\eta}}$$

(stated in his Example 4.1). These conditions can be seen to be reformulations of our inequalities (30) and (31) in the case where population growth is constrained to be non-negative ($\varphi \geqslant 0$).

Our contribution goes beyond that of Mitra [17]: in a setting which includes not only population growth, but also population decline, we have

- presented a complete characterization of paths with *constant* (gross and net of population growth) *savings rates* under population growth, and derived closed form solutions for such paths;
- shown the equivalence between efficiency and constant savings rates, on the one hand, and quasi-arithmetic population growth and the social objectives of maximin and classical utilitarianism, on the other hand;
- generalized the literature on the Hartwick rule and its converse, by considering also the case where population growth is non-zero and by including also classical utilitarianism as an objective.

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Appendix A. Proof of Proposition 15

Let $z^* = a^* - b^* = \beta\varphi + (1-\alpha-\beta)\varphi/\eta$, where a^* and b^* are given by (32) and (33). Rewrite equations (41) and (42) as follows:

$$\dot{a}(t) = v(t)f(a(t), z(t)),$$
$$\dot{z}(t) = v(t)g(a(t), z(t)).$$

Then (a^*, z^*) is the (unique) solution to $f(a,z) = g(a,z) = 0$. For (a,z) such that $g(a,z) \neq 0$, define

$$h(a,z) = \frac{f(a,z)}{g(a,z)}.$$

By L'Hôpital's rule, $\lim_{a\to a^*} h(a, z^*)$ exists. Consider the differential equation

$$\frac{da}{dz} = h(a,z).$$

Fix $(a_0, z_0) = (a^*, z^*)$. Solve the differential equation to find a function $\hat{a}(z)$ passing through (a^*, z^*). The function $\hat{a}$ is uniquely determined, and it defines the stable manifold in (a,z) space for $a<1$ and $z>0$ if $\varphi>0$, and $a<1$ and $z<0$ if $\varphi<0$. This stable manifold is invariant with respect to time. A phase diagram analysis is therefore warranted. If the pair $(a(0), z(0))$ of initial values is chosen on the manifold, convergence to (a^*, z^*) occurs. On the other hand, if the pair $(a(0), z(0))$ is chosen above or below the manifold, then $(a(t), z(t)$ diverges. See Fig. 1.

Since the converging path is interior and satisfies the Hotelling rule and the capital transversality condition, it remains to be shown that the pair of initial values can be chosen on the stable manifold such that exact resource exhaustion takes place. For given K_0 and $N(0)$, there exists S_0^* such that (10)–(12) are satisfied when $a(t) = a^*$ and $b(t) = b^*$ for all t. If $S_0 = S_0^*$, then the path stays at (a^*, z^*) and satisfies resource exhaustion by choosing $a(0) = a^*$ and $z(0) = z^*$. Refer to this solution as the *steady state path*, and denote it by $\{q^*(t), c^*(t), k^*(t), r^*(t)\}_{t=0}^{\infty}$.

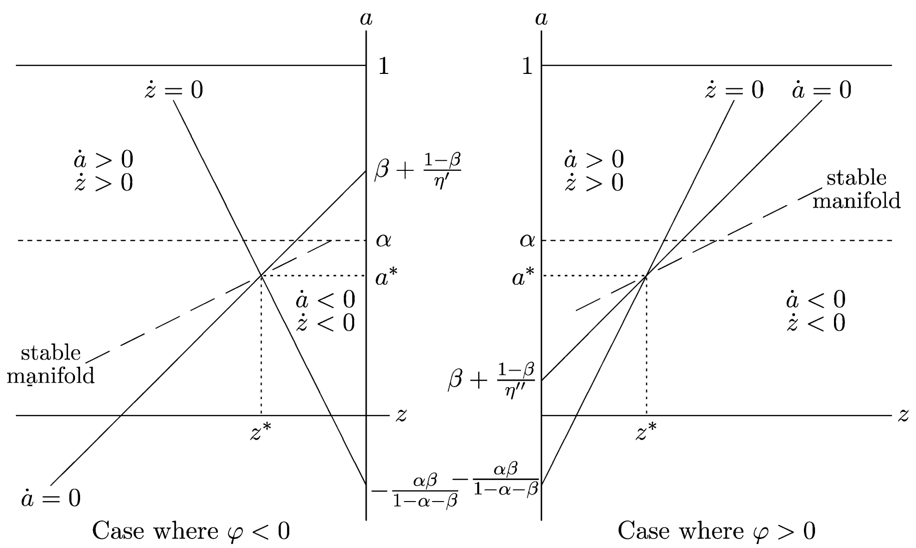

Fig. 1. Phase diagrams for $\varphi<0$ and $\varphi>0$.

If $S_0 \neq S_0^*$, then a converging path satisfies resource exhaustion only if the pair $(a(0), z(0))$ of initial values does not equal (a^*, z^*). In terms of the original variables of the model we can write

$$a(0) = 1 - \frac{c(0)}{q(0)} = 1 - \frac{c(0)}{k(0)^\alpha r(0)^\beta},$$
$$z(0) = v(0)x(0) = v(0)\frac{k(0)}{q(0)} = v(0)\frac{k(0)^{1-\alpha}}{r(0)^\beta},$$

implying that

$$c(0) = v(0)k(0)\frac{1 - a(0)}{z(0)}, \tag{A.1}$$

$$r(0) = \left(v(0)k(0)^{1-\alpha}\frac{1}{z(0)}\right)^{1/\beta}. \tag{A.2}$$

Furthermore, (34) implies that

$$\frac{\dot{c}(t)}{c(t)} = \frac{\alpha v(t)}{\eta}\left(\frac{1}{z(t)}\right), \tag{A.3}$$

while it follows from (22), (15), and (37) that

$$\frac{\dot{r}(t)}{r(t)} = -\frac{\alpha v(t)}{1-\beta}\left(\frac{1 - a(t)}{z(t)} + 1\right). \tag{A.4}$$

Finally, (35), (36), (A.2), and (A.4) imply that total resource extraction is a continuous function of $z(0)$.

Consider first the cases where $\varphi > 0$ and $S_0 \neq S_0^*$.

Let $S_0 > S_0^*$. Choose $a(0) < a^*$ and $z(0) < z^*$ on the stable manifold leading to (a^*, z^*) (i.e., $a(0) = \hat{a}(z(0))$). By (A.1), initial consumption can be made arbitrarily large by choosing $z(0)$ sufficiently small. Since, by (A.3), consumption grows at least as fast as in the steady state, total resource extraction can be made arbitrarily large by choosing $z(0)$ sufficiently small. Because total resource extraction is a continuous function of $z(0)$, it follows that there exists a pair $(a(0), z(0))$ on the stable manifold, with $a(0) < a^*$ and $z(0) < z^*$, such that exact exhaustion of S_0 takes place.

The case where $\varphi > 0$ and $S_0 < S_0^*$ is analogous, since, by (A.1), $c(0)$ can be made arbitrarily small by choosing $z(0)$ sufficiently large.

Consider next the cases where $\varphi < 0$ and $S_0 \neq S_0^*$. In these cases, $z < 0$.

Let $S_0 > S_0^*$. Since $\eta = \infty$ or, by (30) and (31), $\eta > 1$, it follows that $\beta + (1-\beta)/\eta < 1$. Hence, the stable manifold has the property that

$$\lim_{|z| \to 0} \hat{a}(z) \leqslant \beta + \frac{1-\beta}{\eta} < 1; \tag{A.5}$$

see the left panel of Fig. 1. Choose $a(0) < a^*$ and $|z(0)| < |z^*|$ on the stable manifold leading to (a^*, z^*) (i.e., $a(0) = \hat{a}(z(0))$). By (A.1) and (A.5), initial consumption can be made arbitrarily large by choosing $|z(0)|$ sufficiently small. Hence, the argument above when $\varphi > 0$ goes through.

The case where $\varphi < 0$ and $S_0 < S_0^*$ is analogous, provided that we can show that

$$\lim_{|z| \to \infty} \frac{1 - \hat{a}(z)}{|z|} = 0, \tag{A.6}$$

since then, by (A.1), $c(0)$ can be made arbitrarily small by choosing $|z(0)|$ sufficiently large. This can be shown under $-1 \leqslant \varphi < 0$ (implying by (30) and (31) that $\eta > 1$) by transforming (35) and (36) to

$$\frac{d}{dt}\left(\ln\frac{1 - a(t)}{z(t)}\right) = -\frac{\dot{a}(t)}{1 - a(t)} - \frac{\dot{z}(t)}{z(t)} = v(t)\left(\frac{1 - a(t)}{z(t)} - \left(1 - \frac{\alpha}{\eta}\right)\frac{1}{z(t)} + \left(1 + \frac{1}{\varphi}\right)\right), \tag{A.7}$$

$$\frac{d}{dt}\left(\ln\frac{1}{z(t)}\right) = -\frac{\dot{z}(t)}{zt)} = v(t)\left(\frac{1-\alpha-\beta}{1-\beta}\frac{1 - a(t)}{z(t)} - (1-\alpha)\frac{1}{z(t)} + \left(\frac{1-\alpha-\beta}{1-\beta} + \frac{1}{\varphi}\right)\right). \tag{A.8}$$

For suppose that (A.6) does not hold, i.e., $\limsup_{|z|\to\infty}(1-\hat{a}(z))/|z| \geqslant \varepsilon > 0$. Then, using (A.7) and (A.8), it can be shown that there exists a sufficiently large $|z(0)|$ such that the path with $(\hat{a}(z(0), z(0)))$ as initial values satisfies

$$\frac{1-a(t)}{|z(t)|} > \frac{1-a^*}{|z^*|} \quad \text{and} \quad \frac{d}{dt}\left(\frac{1-a(t)}{|z(t)|}\right) > 0$$

for all t beyond some $T \geqslant 0$. This contradicts that, by definition of the function $\hat{a}$, any path with $(\hat{a}(z(0), z(0)))$ as initial values converges to (a^*, z^*).

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